Lecture 12

Mathematical Induction

The Idea

Suppose we know the following:

It’s almost midnight with a clear sky and it has not rained all day.

If it does not rain on some day, then it will also not rain the following day.

Can we conclude then that it will never rain again? Yes.

Mathematical Induction

Principle of Mathematical Induction:

To prove $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, we perform two steps:

Basis Step: Prove that $P(1)$ is true.

Inductive Step: We prove that $P(k) \rightarrow P(k+1)$ is true for all positive integers k.

How do we perform the inductive step?

Assume $P(k)$ is true for any arbitrary positive integer k, and show that if $P(k)$ is true, then $P(k+1)$ is also true. The assumption that $P(k)$ is true is called the **inductive hypothesis**.

Note: Mathematical Induction can also be applied to prove statements on other domains, mathematical objects such as Graphs, or proof correctness of algorithms, etc.

Examples: Mathematical Induction

Theorem: For all positive integers n,

$$1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Proof: Let $P(n)$ be the proportional function that $1^2 + 2^2 + \ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

Basis Step: $P(1)$ is true because $1^2 = \dfrac{1.2.3}{6}$.

Inductive Step: We assume that $P(k)$ is true for an arbitrary positive integer k, i.e.,

$$1^2 + 2^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

continue…

Examples: Mathematical Induction

Under this assumption, we will show that $P(k+1)$ is true as well, i.e.,

$$1^2 + 2^2 + \ldots + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

We start by adding $(k+1)^2$ to both sides of the inductive hypothesis (IH).

$$1^2 + 2^2 + \ldots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= \frac{k(k+1)(2k+1) + 6.(k+1)^2}{6}$$

continue…

Examples: Mathematical Induction

$$1^2 + 2^2 + \ldots + k^2 + (k+1)^2 = \frac{(k+1)(k(2k+1) + 6.(k+1))}{6}$$

$$= \frac{(k+1)(2k^2 + 7k + 6)}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

Thus, $P(k+1)$ is true under the assumption that $P(k)$ is true.

We have completed the basis step and the inductive step, so by mathematical induction we know that $P(n)$ is true for all positive integers n.

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The Good and the Bad of Math. Induction

The Good: It is a good verifier of a claim.

The Bad: It does not give enough intuition behind "why" something is true.

Can you **find** the closed form for $1^3 + 2^3 + \ldots + n^3$ using the proof of previous theorem?

Seems difficult, but if we have the closed form then we can verify whether it's correct.

Some other proof might have given us intuition to find a closed form for $1^3 + 2^3 + \ldots + n^3$.

Examples: Mathematical Induction

Example: Conjecture a formula for the sum of the first n positive odd integers. Then prove the conjecture using mathematical induction.

Solution: The sum of first n positive odd integers for $n = 1, 2, 3, 4,$ and 5 are:

$$1 = 1$$

$$1 + 3 = 4$$

$$1 + 3 + 5 = 9$$

$$1 + 3 + 5 + 7 = 16$$

$$1 + 3 + 5 + 7 + 9 = 25$$

$1 + 3 + \ldots + 2n - 1 = n^2$ looks like a reasonable conjecture.

continue…

Examples: Mathematical Induction

We will now prove $P(n) = 1 + 3 + \ldots + 2n - 1 = n^2$ for all positive n using induction.

Basis Step: $P(1)$ is true because $1 = 1^2$.

Inductive Step: We assume that $P(k)$ is true for an arbitrary positive integer k, i.e.,

$$1 + 3 + \ldots + 2k - 1 = k^2$$

Under this assumption, we will show that $P(k+1)$ is true as well, i.e.,

$$1 + 3 + \ldots + 2k + 1 = (k+1)^2$$

We start by adding $(2k+1)$ to both sides of the IH.

$$1 + 3 + \ldots + 2k - 1 + 2k + 1 = k^2 + 2k + 1$$

$$1 + 3 + \ldots + 2k + 1 = (k+1)^2$$

Thus, $P(k+1)$ is true under the assumption that $P(k)$ is true. ∎

Examples: Mathematical Induction

Example: At a tennis tournament, every two players played against each other exactly one time. After all games were over, each player listed the names of those he defeated, and the names of those defeated by someone he defeated. Prove that at least one player listed the names of everybody else.

Suppose the tournament had 4 players and the following are the results of all matches.

- 1 won the match between 1 and 2.
- 3 won the match between 1 and 3.
- 1 won the match between 1 and 4.
- 2 won the match between 2 and 3.
- 2 won the match between 2 and 4.
- 4 won the match between 3 and 4.

List of player 1: $2, 4, 3$

List of player 2: $3, 4, 1$

List of player 3: $1, 2, 4$

List of player 4: $3, 1$

Examples: Mathematical Induction

Example: At a tennis tournament, every two players played against each other exactly one time. After all games were over, each player listed the names of those he defeated, and the names of those defeated by someone he defeated. Prove that at least one player listed the names of everybody else.

Solution: We will prove the statement for n player tournament, where n is an integer ≥ 2.

Basis Step: For $n = 2$, the statement is trivially true because winner of the only match will list the name of the loser.

Inductive Step: Assume that the statement is true for a k-players tournament.

Under this assumption we will prove the statement for $(k+1)$-players tournament.

Let A be one of the players with least number of victories in a $(k+1)$-players tournament.

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